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Abstract

I find the proofs in *Stochastic Calculus for Finance* to be incredibly dense. The notation can be difficult to follow and each proof calls upon layers of previous, equally-dense results. To ensure my own comprehension, I wanted compose a more thorough treatment of the significant results of Chapter 2.

Definition 0.1 (Expected Value)**.** Let X be a random variable defined on a finite probability space $(\Omega, \mathbb{P})$, where a random variable is a real-valued function defined on Ω, the space of all possible outcomes of some random experiment. The expected value of X is defined to be

$$\mathbb{E}X = \sum_{w \in \Omega} X(w)\mathbb{P}(w)$$

In essence, the expected value is the summation of the random variable value times it's probability for all outcomes, w within Ω.

We may use the *risk-neutral probability measure* $\mathbb{P}$, so it is helpful to recall the motivation behind the risk-neutral probability measure.

Definition 0.2 (Risk-Neutral Probability Measure)**.** Take a simple binomial model of the future price of some derivative. Let X_0 be the starting wealth to be invested in the derivative and Δ_0 be the number of shares purchased at time zero. So at time 0, we have $X_0 - \Delta_0 S_0$ remaining cash, where S_0 is the stock price at time 0. Let r be the money market interest rate at which we invest our cash. So at time one, the value of our portfolio is

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0)$$

To determine the price V_0 of the derivative, we want to find values of X_0, Δ_0 such that $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$. So by dividing the above by $(r+1)$, we want to find

$$X_1 = (1+r)X_0 + \Delta_0(S_1(H) - (1+r)S_0) = V_1(H) \Rightarrow X_0 + \Delta_0(\frac{1}{1+r}S_1(H) - S_0) = \frac{1}{1+r}V_1(H)$$

Similarly, we want

$$X_0 + \Delta_0(\frac{1}{1+r}S_1(T) - S_0) = \frac{1}{1+r}V_1(T)$$

To solve the two equations, we will multiply the first by $\tilde{p}$ and the second by $\tilde{q} = 1 - \tilde{p}$, then add the two equations together.

$$X_0 + \Delta_0(\frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] - S_0) = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]$$

Choose $\tilde{p}$ such that

$$S_0 = \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)]$$

and we see the right hand side of the of the added together equations simplifies to

$$X_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]$$

Remember that in a binomial model, $S_1(H) = uS_0$ and $S_1(T) = dS_0$ where u, d are the factor by which the stock price increases or decreases depending on a head or tails. So

$$S_0 = \frac{1}{1+r}[\tilde{p}uS_0 + (1-\tilde{p})dS_0] = \frac{S_0}{1+r}[(u-d)\tilde{p} + d]$$

Solving for $\tilde{p}$, we get $\tilde{p} = \frac{1+r-d}{u-d}$. Similarly, $\tilde{q} = \frac{u-1-r}{u-d}$

These values are referred to as the *risk neutral probabilities* because they allow us to perfectly replicate the performance of a derivative and find the arbitrage-free price of the derivative.

Definition 0.3 (Conditional Expectation)**.** Now that we have the risk-neutral probabilities, we see that the stock price at time n is equal to

$$S_n(w_1...w_n) = \frac{1}{1+r}[\tilde{p}S_{n+1}(w_1...w_nH) + \tilde{q}S_{n+1}(w_1...w_nT)]$$

Under the risk-neutral probability measure on a binomial pricing model,

$$\tilde{\mathbb{E}}S_{n+1}(w_1....w_n) = \sum_{w\in\Omega} S_{n+1}(w_1....w_n)\tilde{\mathbb{P}}(w) = \tilde{p}S_{n+1}(w_1...w_nH) + \tilde{q}S_{n+1}(w_1...w_nT)$$

So we can rewrite the stock price as

$$S_n(w_1...w_n) = \frac{1}{1+r}\tilde{\mathbb{E}}_nS_{n+1}(w_1...w_n)$$

This is called the *conditional expectation* of S_{n+1}

Definition 0.4 (discounted asset)**.** A "discounted asset" is simply an asset denominated in another asset. By no arbitrage, any asset denominated by another asset is a martingale under the measure induced by the denominating asset (Theorem 2.4.4)

Definition 0.5 (Martingales)**.** Taking the previous equation, dividing by $(1+r)^n$ gives

$$\frac{S_n}{(1+r)^n} = \tilde{\mathbb{E}}_n[\frac{S_{n+1}}{(1+r)^{n+1}}]$$

This equation shows that the conditional expectation of the discounted stock price ($\tilde{\mathbb{E}}_n\frac{S_{n+1}}{(1+r)^{n+1}}$) at time $n+1$ is the discounted price of at time n. Processes that satisfy this condition are called *martingales*.

In general, for some sequence of random variables $M_0...M_n$, a process is a martingale if

$$M_n = \mathbb{E}_n[M_{n+1}], \forall n$$

In essence, a martingale is a process whose expected value remains constant through all time steps.

Definition 0.6 (Markov Process)**.** Consider the binomial asset-pricing model. Let $X_0, X_1, ..., X_N$ be an adapted process. If, for every n between 0 and $N-1$ and for every $f(x)$ there exists a function $g(x)$ (depending on n and f) such that

$$\mathbb{E}_n[f(X_{n+1})] = g(X_n)$$

then $X_0, X_1, ..., X_n$ is a Markov process.

By the definition of the left hand side expected value, we must know the result of the first n coin tosses in order to evaluate the value. If there exists some function $g(x)$ as described, we need only know the value of X_n to determine the expected value. Thus, the existence of a function $g(x)$ proves a significant computational advantage.

Problem 1 (2.1)**.** Using Definition 2.1.1, show the following:
(i) If A is an event and A^c denotes its complement, then $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
(ii) If $A_1, ..., A_N$ is a finite set of events, then

$$\mathbb{P}(\bigcup_{n=1}^{N} A_n) \leq \sum_{n=1}^{N} \mathbb{P}(A_n)$$

If the events $A_1, ..., A_N$ are disjoint, then equality holds for the above.

Solution. (i) We begin by recalling definition 2.1.1:
A finite probability space consists of a sample space Ω and a probability measure $\mathbb{P}$. The sample space is a nonempty finite set and a probability measure is a function that assigns to each element $w \in \Omega$ a number in $[0, 1]$ so that

$$\sum_{x \in \Omega} \mathbb{P}(w) = 1$$

An event is a subset of Ω, and we define the probability of an event A to be

$$\mathbb{P}(A) = \sum_{w \in A} \mathbb{P}(w)$$

Let A be an event by the above definition. So $A \subseteq \Omega$ and

$$\mathbb{P}(A) = \sum_{w \in A} \mathbb{P}(w)$$

Now since A^c is the complement of A, we know that for all w in A, $w \notin A^c$ and the reverse holds. So A and A^c are disjoint and since a set and its complement are equal to the probability space, and the probability of the probability space is one by definition,

$$\mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c) = 1 \Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

(ii) Let $A_1, ..., A_N$ be a finite set of events. In the case they are disjoint, then

$$\mathbb{P}(A_1 \cup ... \cup A_N) = \sum_{w \in A_1 \ cup ... \cup A_N} \mathbb{P}(w) = \sum_{w \in A_1} \mathbb{P}(w) + ... \sum_{w \in A_N} \mathbb{P}(w) = \mathbb{P}(A_1) + ... + \mathbb{P}(A_N)$$

If not disjoint, then

$$\mathbb{P}(A_1 \cup ... \cup A_N) = \mathbb{P}(A_1 - ... - A_N) \cup ... \cup A_N)$$

$$= \mathbb{P}(A_1 - ... - A_N) + \mathbb{P}(A_2 - A_1 - ... - A_N) + ... \leq \mathbb{P}(A_1) + ... + \mathbb{P}(A_N) = \sum_{n=1}^{N} \mathbb{P}(A_n)$$

□

Problem 2 (2.2)**.** Consider the stock price S_3 in Figure 2.3.1.
(i) What is the distribution of S_3 under the risk-neutral probabilities $\tilde{p} = 1/2, \tilde{q} = 1/2$.
(ii) Compute $\tilde{E}S_1, \tilde{E}S_2, \tilde{E}S_3$. What is the average rate of growth of the stock price under $\tilde{P}$?
(iii) Answer (i) and (ii) again under the actual probabilities $p = 2/3, q = 1/3$.

Solution.
(i)
$\tilde{P}(S_3(HHH)) = 1/8$
$\tilde{P}(S_3(HHT, HTH, THH)) = 3/8$
$\tilde{P}(S_3(HTT, THT, TTH)) = 3/8$
$\tilde{P}(S_3(TTT)) = 1/8$
(ii)
From Definition 2.3.1,

$$\tilde{\mathbb{E}}S_1 = \sum_{w_{n+1}...w_N} \tilde{p}^{\#H}\tilde{q}^{\#T}X = (1/2)^1(1/2)^0 8 + (1/2)^0(1/2)^1 2 = 4 + 1 = 5$$

$$\tilde{\mathbb{E}}S_2 = \sum_{w_{n+1}...w_N} \tilde{p}^{\#H}\tilde{q}^{\#T}X = (1/2)^2(1/2)^0 16 + (1/2)^1(1/2)^1 4 + (1/2)^1(1/2)^1 4 + (1/2)^0(1/2)^2 1$$

$$= 4 + 1 + 1 + .25 = 6.25$$

$$\tilde{\mathbb{E}}S_3 = \sum_{w_{n+1}...w_N} \tilde{p}^{\#H}\tilde{q}^{\#T}X = (1/2)^3(1/2)^0 32 + (1/2)^2(1/2)^1 8 + (1/2)^2(1/2)^1 8 + (1/2)^2(1/2)^1 8 + (1/2)^1(1/2)^2 2$$

$$+ (1/2)^1(1/2)^2 2 + (1/2)^1(1/2)^2 2 + (1/2)^0(1/2)^3 .5 = 4 + 1 + 1 + 1 + .25 + .25 + .25 + .0625 = 7.8125$$

So the average rates of growth are

$$r_0 = (6 - 4)/4 = 1/2, r_1 = (6.25 - 5)/5 = .25, r_3 = (7.8125 - 6.25)/6.25 = .25$$

(iii)
$\tilde{P}(S_3(HHH)) = 8/27$
$\tilde{P}(S_3(HHT, HTH, THH)) = 4/9$
$\tilde{P}(S_3(HTT, TTH, THT)) = 2/9$
$\tilde{P}(S_3(TTT)) = 1/27$
Then

$$\tilde{\mathbb{E}}S_1 = \sum_{w_{n+1}...w_N} \tilde{p}^{\#H}\tilde{q}^{\#T}X = (2/3)^1(1/3)^0 8 + (2/3)^0(1/3)^1 2 = 16/3 + 2/3 = 18/3 = 6$$

$$\tilde{\mathbb{E}}S_2 = \sum_{w_{n+1}\dots w_N} \tilde{p}^{\#H}\tilde{q}^{\#T}X = (2/3)^2(1/3)^0 16{+}2{*}(2/3)(1/3)4{+}(2/3)^0(1/3)^2 1 = 64/9{+}16/9{+}1/9 = 9$$

$$\tilde{\mathbb{E}}S_3 = \sum_{w_{n+1}\dots w_N} \tilde{p}^{\#H}\tilde{q}^{\#T}X = (2/3)^3(1/3)^0 32{+}3{*}(2/3)^2(1/3)^1 8{+}3{*}(2/3)^1(1/3)^2 2{+}(2/3)^0(1/3)^3.5$$

$$= 256/27 + 96/27 + 8/27 + 1/27 = 361/27 \approx 13.37$$

Then the average rates of growth are

$$r_0 = (6-4)/6 = .5, r_1 = (9-6)/6 = .5, r_2 = (13.5-9)/9 = .5$$

□

Problem 3 (2.3)**.** Show that a convex function of a martingale is a submartingale. In other words, let $M_0, M_1, ..., M_N$ be a martingale and let φ be a convex function. Show that $\varphi(M_0), ..., \varphi(M_N)$ is a submartingale.

Solution. By Jensen's Inequality, we know that

$$\mathbb{E}[\varphi(X)] \geq \varphi(\mathbb{E}X)$$

Since $M_0, ..., M_N$ is a martingale,

$$M_n = \mathbb{E}_n[M_{n+1}]$$

Then by substitution and the fundamental properties of conditional expectations,

$$\varphi(\mathbb{E}_n[M_{n+1}]) = \varphi(M_n) \leq \mathbb{E}_n\varphi[M_{n+1}]$$

So φ is a submartingale by definition. □

Problem 4 (2.10 (Dividend-paying stock))**.**

Solution. (i) Take the risk-neutral conditional expectation of the defined wealth process

$$\tilde{\mathbb{E}_n}[\frac{X_{n+1}}{(1+r)^{n+1}}]$$

To show the dividend-paying wealth process is a martingale, we must show that the conditional expectation equals

$$\frac{X_n}{(1+r)^n}$$

So by substituting in the definition of the wealth process,

$$\tilde{\mathbb{E}_n}[\frac{X_{n+1}}{(1+r)^{n+1}}] = \tilde{\mathbb{E}_n}[\frac{\Delta_n Y_{n+1}S_n + (1+r)(X_n - \Delta_n S_n)}{(1+r)^{n+1}}] = \tilde{\mathbb{E}_n}[\frac{\Delta_n Y_{n+1}S_n}{(1+r)^{n+1}} + \frac{(1+r)(X_n - \Delta_n S_n)}{(1+r)^{n+1}}]$$

$$= \frac{\Delta_n S_n}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n[Y_{n+1}] + \frac{X_n - \Delta_n S_n}{(1+r)^n} = \frac{\Delta_n S_n}{(1+r)^{n+1}}(u\tilde{p} + d\tilde{q}) + \frac{X_n - \Delta_n S_n}{(1+r)^n}$$

$$= \frac{\Delta_n S_n + X_n - \Delta_n S_n}{(1+r)^n} = \frac{X_n}{(1+r)^n}$$

(ii) Next we show that the risk-neutral pricing formula still applies (i.e. Theorem 2.4.7 holds for the dividend-paying model). Using the definition of the wealth process we see that

$$\Delta_n = \frac{X_{n+1}(H) - X_{n+1}(T)}{uS_n - dS_n}$$

and

$$X_n = \tilde{\mathbb{E}}_n[\frac{X_{n+1}}{1+r}]$$

Since our goal is to replicate the payoff at time N, we set $X_N = V_N$. So

$$V_n = X_n = \tilde{\mathbb{E}}_n[\frac{V_N}{(1+r)^{N-n}}]$$

□